Pullback Attractors for Non-autonomous Reaction-Diffusion Equations on $\mathbb{R}^n$

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Abstract

We study the long time behavior of solutions of the non-autonomous Reaction-Diffusion equation defined on the entire space $\mathbb{R}^n$ when external terms are unbounded in a phase space. The existence of a pullback global attractor for the equation is established in $L^2(\mathbb{R}^n)$ and $H^1(\mathbb{R}^n)$, respectively. The pullback asymptotic compactness of solutions is proved by using uniform a priori estimates on the tails of solutions outside bounded domains.

Key words. pullback attractor, asymptotic compactness, non-autonomous equation.

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1 Introduction

In this paper, we study the dynamical behavior of the non-autonomous Reaction-Diffusion equation defined on $\mathbb{R}^n$:

$$\frac{\partial u}{\partial t} - \Delta u + \lambda u = f(x,u) + g(x,t), \tag{1.1}$$

where λ is a positive constant, g is a given function in $L^2_{loc}(\mathbb{R}, L^2(\mathbb{R}^n))$, and f is a nonlinear function satisfying a dissipative condition.

Global attractors for non-autonomous dynamical systems have been extensively studied in the literature, see, e.g., [1, 3, 8, 10, 11, 12, 13, 14, 15, 16, 20, 22, 23, 25, 26, 28, 30, 31, 34, 39]. Particularly, when PDEs are defined in bounded domains, such attractors have been investigated

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in [8, 12, 13, 14, 16, 20, 25, 30, 31]. In the case of unbounded domains, global attractors for non-autonomous PDEs have been examined in [1, 26, 28] for almost periodic external terms, and in [10, 11, 34, 39] for unbounded external terms. In this paper, we will prove existence of a pullback attractor for equation (1.1) defined on $\mathbb{R}^n$ with unbounded external terms.

Notice that the domain $\mathbb{R}^n$ for (1.1) is unbounded, and hence Sobolev embeddings are no longer compact in this case. This introduces a major obstacle for examining the asymptotic compactness of solutions. For some PDEs, such difficulty can be overcome by the energy equation approach, which was introduced by Ball in [4, 5] (see also [10, 11, 19, 21, 24, 26, 27, 29, 37, 38]). In this paper, we will use the uniform estimates on the tails of solutions to circumvent the difficulty caused by the unboundedness of the domain. This idea was developed in [33] to prove asymptotic compactness of solutions for autonomous parabolic equations on $\mathbb{R}^n$, and later extended to non-autonomous equations in [1, 28, 34, 39] and stochastic equations in [7, 35, 36]. Here, we will use the method of tail-estimates to investigate the asymptotic behavior of equation (1.1) with nonlinearity of arbitrary growth rate. We first establish the pullback asymptotic compactness of solutions of equation (1.1) and prove existence of a pullback global attractor in $L^2(\mathbb{R}^n)$. Then we extend this result and show existence of a pullback global attractor in $H^1(\mathbb{R}^n)$.

It is worth noticing that attractors for the non-autonomous Reaction-Diffusion equation defined on $\mathbb{R}^n$ with unbounded external terms were also studied in [39], where the authors proved the existence of a pullback attractor when the nonlinearity f satisfies a Sobolev growth rate. In the present paper, we deal with the case where the growth order of f is arbitrary. The asymptotic compactness of solutions in [39] was obtained by using the energy equation approach. But, here we will derive such compactness directly from the uniform tail-estimates of solutions. As we will see later, the existence of an attractor in $H^1(\mathbb{R}^n)$ is an immediate consequence of the existence of an attractor in $L^2(\mathbb{R}^n)$ and the asymptotic compactness of solutions in $H^1(\mathbb{R}^n)$.

The paper is organized as follows. In the next section, we recall fundamental concepts and results for pullback attractors for non-autonomous dynamical systems. In Section 3, we define a cocycle for the non-autonomous Reaction-Diffusion equation on $\mathbb{R}^n$. Section 4 is devoted to deriving uniform estimates of solutions for large space and time variables. In the last section, we prove the existence of a pullback global attractor for the equation in $L^2(\mathbb{R}^n)$ and $H^1(\mathbb{R}^n)$.

The following notations will be used throughout the paper. We denote by $\|\cdot\|$ and $(\cdot,\cdot)$ the norm

and inner product in $L^2(\mathbb{R}^n)$ and use $\|\cdot\|_p$ to denote the norm in $L^p(\mathbb{R}^n)$. Otherwise, the norm of a general Banach space X is written as $\|\cdot\|_X$. The letters C and C_i $(i = 1, 2, \dots)$ are generic positive constants which may change their values from line to line or even in the same line.

2 Preliminaries

In this section, we recall some basic concepts related to pullback attractors for non-autonomous dynamical systems. It is worth noticing that these concepts are quite similar to that of random attractor for stochastic systems. We refer the reader to [2, 6, 7, 9, 10, 11, 13, 17, 18, 30, 35] for more details.

Let Ω be a nonempty set and X a metric space with distance $d(\cdot,\cdot)$.

Definition 2.1. A family of mappings $\{\theta_t\}_{t\in\mathbb{R}}$ from Ω to itself is called a family of shift operators on Ω if $\{\theta_t\}_{t\in\mathbb{R}}$ satisfies the group properties:

(i) $\theta_0\omega = \omega, \quad \forall\ \omega \in \Omega;$

(ii) $\theta_t(\theta_\tau\omega) = \theta_{t+\tau}\omega, \quad \forall\ \omega \in \Omega \quad \text{and} \quad t,\ \tau \in \mathbb{R}.$

Definition 2.2. Let $\{\theta_t\}_{t\in\mathbb{R}}$ be a family of shift operators on Ω. Then a continuous θ-cocycle ϕ on X is a mapping

$$\phi : \mathbb{R}^+ \times \Omega \times X \to X, \quad (t, \omega, x) \mapsto \phi(t, \omega, x),$$

which satisfies, for all $\omega \in \Omega$ and $t, \tau \in \mathbb{R}^+$,

(i) $\phi(0, \omega, \cdot)$ is the identity on X;

(ii) $\phi(t + \tau, \omega, \cdot) = \phi(t, \theta_\tau\omega, \cdot) \circ \phi(\tau, \omega, \cdot);$

(iii) $\phi(t, \omega, \cdot) : X \to X$ is continuous.

Hereafter, we always assume that ϕ is a continuous θ-cocycle on X, and $\mathcal{D}$ a collection of families of subsets of X:

$$\mathcal{D} = \{D = \{D(\omega)\}_{\omega\in\Omega} :\ D(\omega) \subseteq X \text{ for every } \omega \in \Omega\}.$$

Such a collection $\mathcal{D}$ is often referred to as a universe in the literature.

Definition 2.3. Let $\mathcal{D}$ be a collection of families of subsets of X. Then $\mathcal{D}$ is called inclusion-closed if $D = \{D(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$ and $\tilde{D} = \{\tilde{D}(\omega) \subseteq X : \omega \in \Omega\}$ with $\tilde{D}(\omega) \subseteq D(\omega)$ for all $\omega \in \Omega$ imply that $\tilde{D} \in \mathcal{D}$.

Definition 2.4. Let $\mathcal{D}$ be a collection of families of subsets of X and $\{K(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$. Then $\{K(\omega)\}_{\omega\in\Omega}$ is called a pullback absorbing set for ϕ in $\mathcal{D}$ if for every $B \in \mathcal{D}$ and $\omega \in \Omega$, there exists $t(\omega, B) > 0$ such that

$$\phi(t, \theta_{-t}\omega, B(\theta_{-t}\omega)) \subseteq K(\omega) \quad \text{for all } t \geq t(\omega, B).$$

Definition 2.5. Let $\mathcal{D}$ be a collection of families of subsets of X. Then ϕ is said to be $\mathcal{D}$-pullback asymptotically compact in X if for every $\omega \in \Omega$, $\{\phi(t_n, \theta_{-t_n}\omega, x_n)\}_{n=1}^{\infty}$ has a convergent subsequence in X whenever $t_n \to \infty$, and $x_n \in B(\theta_{-t_n}\omega)$ with $\{B(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$.

Definition 2.6. Let $\mathcal{D}$ be a collection of families of subsets of X and $\{\mathcal{A}(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$. Then $\{\mathcal{A}(\omega)\}_{\omega\in\Omega}$ is called a $\mathcal{D}$-pullback global attractor for ϕ if the following conditions are satisfied, for every $\omega \in \Omega$,

(i) $\mathcal{A}(\omega)$ is compact;

(ii) $\{\mathcal{A}(\omega)\}_{\omega\in\Omega}$ is invariant, that is,

$$\phi(t, \omega, \mathcal{A}(\omega)) = \mathcal{A}(\theta_t\omega), \quad \forall\, t \geq 0;$$

(iii) $\{\mathcal{A}(\omega)\}_{\omega\in\Omega}$ attracts every set in $\mathcal{D}$, that is, for every $B = \{B(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$,

$$\lim_{t\to\infty} d(\phi(t, \theta_{-t}\omega, B(\theta_{-t}\omega)), \mathcal{A}(\omega)) = 0,$$

where d is the Hausdorff semi-metric given by $d(Y, Z) = \sup_{y\in Y} \inf_{z\in Z} \|y - z\|_X$ for any $Y \subseteq X$ and $Z \subseteq X$.

The following existence result of a pullback global attractor for a continuous cocycle can be found in [2, 6, 7, 9, 10, 11, 13, 17, 18].

Proposition 2.7. *Let $\mathcal{D}$ be an inclusion-closed collection of families of subsets of X and ϕ a continuous θ-cocycle on X. Suppose that $\{K(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$ is a closed absorbing set for ϕ in $\mathcal{D}$ and ϕ is $\mathcal{D}$-pullback asymptotically compact in X. Then ϕ has a unique $\mathcal{D}$-pullback global attractor $\{\mathcal{A}(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$ which is given by*

$$\mathcal{A}(\omega) = \bigcap_{\tau\geq 0} \overline{\bigcup_{t\geq\tau} \phi(t, \theta_{-t}\omega, K(\theta_{-t}\omega))}.$$

3 Cocycle associated with the Reaction-Diffusion equation

In this section, we construct a θ-cocycle ϕ for the non-autonomous Reaction-Diffusion equation defined on $\mathbb{R}^n$. For every $\tau \in \mathbb{R}$ and $t > \tau$, consider the problem:

$$\frac{\partial u}{\partial t} - \Delta u + \lambda u = f(x, u) + g(x, t), \quad x \in \mathbb{R}^n, \tag{3.1}$$

with the initial condition

$$u(x, \tau) = u_\tau(x), \quad x \in \mathbb{R}^n, \tag{3.2}$$

where λ is a positive constant, g is given in $L^2_{loc}(\mathbb{R}, L^2(\mathbb{R}^n))$, and f is a nonlinear function satisfying, for every $x \in \mathbb{R}^n$ and $s \in \mathbb{R}$,

$$f(x, s)s \leq -\alpha_1 |s|^p + \phi_1(x) \quad \text{for some} \ \ p \geq 2, \tag{3.3}$$

$$|f(x, s)| \leq \alpha_2 |s|^{p-1} + \phi_2(x), \tag{3.4}$$

$$\frac{\partial f}{\partial s}(x, s) \leq \alpha_3, \tag{3.5}$$

where α_1, α_2 and α_3 are all positive constants, $\phi_1 \in L^1(\mathbb{R}^n)$, and $\phi_2 \in L^2(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)$ with $\frac{1}{p} + \frac{1}{q} = 1$. Denote by $F(x, s) = \int_0^s f(x, \tau) d\tau$. Then we assume that F satisfies

$$-\phi_4(x) - \alpha_4 |s|^p \leq F(x, s) \leq -\alpha_5 |s|^p + \phi_3(x), \tag{3.6}$$

where α_4 and α_5 are positive constants and $\phi_3, \phi_4 \in L^1(\mathbb{R}^n)$.

As in the case of bounded domains (see, e.g., [32]), it can be proved that if $g \in L^2_{loc}(\mathbb{R}, L^2(\mathbb{R}^n))$ and (3.3)-(3.6) hold true, then problem (3.1)-(3.2) is well-posed in $L^2(\mathbb{R}^n)$, that is, for every $\tau \in \mathbb{R}$ and $u_\tau \in L^2(\mathbb{R}^n)$, there exists a unique solution $u \in C([\tau, \infty), L^2(\mathbb{R}^n)) \bigcap L^2(\tau, \tau+T; H^1(\mathbb{R}^n)) \bigcap L^p(\tau, \tau+T; L^p(\mathbb{R}^n))$ for every $T > 0$. Further, the solution is continuous with respect to u_τ in $L^2(\mathbb{R}^n)$. To construct a cocycle ϕ for problem (3.1)-(3.2), we denote by $\Omega = \mathbb{R}$, and define a shift operator θ_t on Ω for every $t \in \mathbb{R}$ by

$$\theta_t(\tau) = t + \tau, \quad \text{for all} \ \ \tau \in \mathbb{R}.$$

Let ϕ be a mapping from $\mathbb{R}^+ \times \Omega \times L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$ given by

$$\phi(t, \tau, u_\tau) = u(t + \tau, \tau, u_\tau),$$

where $t \geq 0$, $\tau \in \mathbb{R}$, $u_\tau \in L^2(\mathbb{R}^n)$, and u is the solution of problem (3.1)-(3.2). By the uniqueness of solutions, we find that for every $t, s \geq 0$, $\tau \in \mathbb{R}$ and $u_\tau \in L^2(\mathbb{R}^n)$,

$$\phi(t+s, \tau, u_\tau) = \phi(t, s+\tau, (\phi(s, \tau, u_\tau))).$$

Then it follows that ϕ is a continuous θ-cocycle on $L^2(\mathbb{R}^n)$. The purpose of this paper is to study the existence of pullback attractors for ϕ in an appropriate phase space.

Let E be a subset of $L^2(\mathbb{R}^n)$ and denote by

$$\|E\| = \sup_{x \in E} \|x\|_{L^2(\mathbb{R}^n)}.$$

Suppose $D = \{D(t)\}_{t \in \mathbb{R}}$ is a family of subsets of $L^2(\mathbb{R}^n)$ satisfying

$$\lim_{t \to -\infty} e^{\lambda t} \|D(t)\|^2 = 0, \tag{3.7}$$

where λ is the positive constant appearing in (3.1). Hereafter, we use $\mathcal{D}_\lambda$ to denote the collection of all families of subsets of $L^2(\mathbb{R}^n)$ satisfying (3.7), that is,

$$\mathcal{D}_\lambda = \{D = \{D(t)\}_{t \in \mathbb{R}} : D \text{ satisfies (3.7)}\}. \tag{3.8}$$

Throughout this paper, we assume the following conditions for the external term:

$$\int_{-\infty}^{\tau} e^{\lambda \xi} \|g(\xi)\|^2 d\xi < \infty, \quad \forall\, \tau \in \mathbb{R}, \tag{3.9}$$

and

$$\limsup_{k \to \infty} \int_{-\infty}^{\tau} \int_{|x| \geq k} e^{\lambda \xi} |g(x, \xi)|^2 dx d\xi = 0, \quad \forall\, \tau \in \mathbb{R}. \tag{3.10}$$

We remark that condition (3.9) is useful for proving existence of absorbing sets for problem (3.1)-(3.2), while the asymptotically null condition (3.10) is crucial for establishing the asymptotic compactness of solutions. Notice that conditions (3.9) and (3.10) do not require that g be bounded in $L^2(\mathbb{R}^n)$ when $t \to \pm\infty$. Particularly, These assumptions do not have any restriction on g when $t \to +\infty$.

It follows from (3.10) that for every $\tau \in \mathbb{R}$ and $\eta > 0$, there is $K = K(\tau, \eta) > 0$ such that

$$\int_{-\infty}^{\tau} \int_{|x| \geq K} e^{\lambda \xi} |g(x, \xi)|^2 dx d\xi \leq \eta e^{\lambda \tau}. \tag{3.11}$$

As we will see later, inequality (3.11) is crucial for deriving uniform estimates on the tails of solutions and these estimates are necessary for proving the asymptotic compactness of solutions.

4 Uniform estimates of solutions

In this section, we derive uniform estimates of solutions of problem (3.1)-(3.2) defined on $\mathbb{R}^n$ when $t \to \infty$. We start with the estimates in $L^2(\mathbb{R}^n)$.

Lemma 4.1. *Suppose* (3.3) *and* (3.9) *hold. Then for every* $\tau \in \mathbb{R}$ *and* $D = \{D(t)\}_{t\in\mathbb{R}} \in \mathcal{D}_\lambda$*, there exists* $T = T(\tau, D) > 0$ *such that for all* $t \geq T$*,*

$$\|u(\tau, \tau - t, u_0(\tau - t))\|^2 \leq M + Me^{-\lambda\tau} \int_{-\infty}^{\tau} e^{\lambda\xi}\|g(\xi)\|^2 d\xi,$$

$$\int_{\tau-t}^{\tau} e^{\lambda\xi}\|u(\xi, \tau - t, u_0(\tau - t))\|_p^p d\xi \leq Me^{\lambda\tau} + M \int_{-\infty}^{\tau} e^{\lambda\xi}\|g(\xi)\|^2 d\xi,$$

and

$$\int_{\tau-t}^{\tau} e^{\lambda\xi}\|u(\xi, \tau - t, u_0(\tau - t))\|_{H^1}^2 d\xi \leq Me^{\lambda\tau} + M \int_{-\infty}^{\tau} e^{\lambda\xi}\|g(\xi)\|^2 d\xi,$$

where $u_0(\tau - t) \in D(\tau - t)$*, and* M *is a positive constant independent of* τ *and* D*.*

Proof. Taking the inner product of (3.1) with u in $L^2(\mathbb{R}^n)$ we get that

$$\frac{1}{2}\frac{d}{dt}\|u\|^2 + \|\nabla u\|^2 + \lambda\|u\|^2 = \int_{\mathbb{R}^n} f(x, u)u dx + (g, u). \tag{4.1}$$

For the nonlinear term, by (3.3) we have

$$\int_{\mathbb{R}^n} f(x, u)u dx \leq -\alpha_1 \int_{\mathbb{R}^n} |u|^p dx + \int_{\mathbb{R}^n} \phi_1 dx. \tag{4.2}$$

By the Young inequality, the last term on the right-hand side of (4.1) is bounded by

$$|(g, u)| \leq \|g\| \ \|u\| \leq \frac{1}{4}\lambda\|u\|^2 + \frac{1}{\lambda}\|g\|^2. \tag{4.3}$$

It follows from (4.1)-(4.3) that

$$\frac{d}{dt}\|u\|^2 + 2\|\nabla u\|^2 + \lambda\|u\|^2 + \frac{1}{2}\lambda\|u\|^2 + 2\alpha_1 \int_{\mathbb{R}^n} |u|^p dx \leq C + \frac{2}{\lambda}\|g\|^2. \tag{4.4}$$

Multiplying (4.4) by $e^{\lambda t}$ and then integrating the resulting inequality on $(\tau - t, \tau)$ with $t \geq 0$, we find that

$$\|u(\tau, \tau - t, u_0(\tau - t))\|^2 + 2e^{-\lambda\tau} \int_{\tau-t}^{\tau} e^{\lambda\xi}\|\nabla u(\xi, \tau - t, u_0(\tau - t))\|^2 d\xi$$

$$+\frac{1}{2}\lambda e^{-\lambda\tau} \int_{\tau-t}^{\tau} e^{\lambda\xi}\|u(\xi, \tau - t, u_0(\tau - t))\|^2 d\xi + 2\alpha_1 e^{-\lambda\tau} \int_{\tau-t}^{\tau} e^{\lambda\xi}\|u(\xi, \tau - t, u_0(\tau - t))\|_p^p d\xi$$

$$\leq e^{-\lambda\tau} e^{\lambda(\tau-t)} \|u_0(\tau-t)\|^2 + \frac{2}{\lambda} e^{-\lambda\tau} \int_{\tau-t}^{\tau} e^{\lambda\xi} \|g(\xi)\|^2 d\xi + \frac{C}{\lambda}$$

$$\leq e^{-\lambda\tau} e^{\lambda(\tau-t)} \|u_0(\tau-t)\|^2 + \frac{2}{\lambda} e^{-\lambda\tau} \int_{-\infty}^{\tau} e^{\lambda\xi} \|g(\xi)\|^2 d\xi + \frac{C}{\lambda}. \tag{4.5}$$

Notice that $u_0(\tau-t) \in D(\tau-t)$ and $D = \{D(t)\}_{t\in\mathbb{R}} \in \mathcal{D}_\lambda$. We find that for every $\tau \in \mathbb{R}$, there exists $T = T(\tau, D)$ such that for all $t \geq T$,

$$e^{\lambda(\tau-t)} \|u_0(\tau-t)\|^2 \leq \frac{1}{\lambda} \int_{-\infty}^{\tau} e^{\lambda\xi} \|g(\xi)\|^2 d\xi. \tag{4.6}$$

By (4.5)-(4.6) we get that, for all $t \geq T$,

$$\|u(\tau, \tau-t, u_0(\tau-t))\|^2 + 2e^{-\lambda\tau} \int_{\tau-t}^{\tau} e^{\lambda\xi} \|\nabla u(\xi, \tau-t, u_0(\tau-t))\|^2 d\xi$$

$$+\frac{1}{2}\lambda e^{-\lambda\tau} \int_{\tau-t}^{\tau} e^{\lambda\xi} \|u(\xi, \tau-t, u_0(\tau-t))\|^2 d\xi + 2\alpha_1 e^{-\lambda\tau} \int_{\tau-t}^{\tau} e^{\lambda\xi} \|u(\xi, \tau-t, u_0(\tau-t))\|_p^p d\xi$$

$$\leq \frac{3}{\lambda} e^{-\lambda\tau} \int_{-\infty}^{\tau} e^{\lambda\xi} \|g(\xi)\|^2 d\xi + \frac{C}{\lambda},$$

which completes the proof. □

The following lemma is useful for deriving uniform estimates of solutions in $H^1(\mathbb{R}^n)$.

Lemma 4.2. *Suppose* (3.3) *and* (3.9) *hold. Then for every $\tau \in \mathbb{R}$ and $D = \{D(t)\}_{t\in\mathbb{R}} \in \mathcal{D}_\lambda$, there exists $T = T(\tau, D) > 2$ such that for all $t \geq T$,*

$$\int_{\tau-2}^{\tau} e^{\lambda\xi} \|u(\xi, \tau-t, u_0(\tau-t))\|^2 d\xi \leq M e^{\lambda\tau} + M \int_{-\infty}^{\tau} e^{\lambda\xi} \|g(\xi)\|^2 d\xi,$$

$$\int_{\tau-2}^{\tau} e^{\lambda\xi} \|\nabla u(\xi, \tau-t, u_0(\tau-t))\|^2 d\xi \leq M e^{\lambda\tau} + M \int_{-\infty}^{\tau} e^{\lambda\xi} \|g(\xi)\|^2 d\xi,$$

and

$$\int_{\tau-2}^{\tau} e^{\lambda\xi} \|u(\xi, \tau-t, u_0(\tau-t))\|_p^p d\xi \leq M e^{\lambda\tau} + M \int_{-\infty}^{\tau} e^{\lambda\xi} \|g(\xi)\|^2 d\xi,$$

where $u_0(\tau-t) \in D(\tau-t)$, and M is a positive constant independent of τ and D.

Proof. By (4.4) we find that

$$\frac{d}{dt}\|u\|^2 + \lambda\|u\|^2 \leq C + \frac{2}{\lambda}\|g\|^2.$$

Let $s \in [\tau-2, \tau]$ and $t \geq 2$. Multiplying the above by $e^{\lambda t}$ and integrating over $(s, \tau-t)$, we get

$$e^{\lambda s} \|u(s, \tau-t, u_0(\tau-t))\|^2 \leq e^{\lambda(\tau-t)} \|u_0(\tau-t)\|^2 + C \int_{\tau-t}^{s} e^{\lambda\xi} d\xi + \frac{2}{\lambda} \int_{\tau-t}^{s} e^{\lambda\xi} \|g(\xi)\|^2 d\xi$$

$$\leq e^{\lambda(\tau-t)}\|u_0(\tau-t)\|^2 + \frac{C}{\lambda}e^{\lambda\tau} + \frac{2}{\lambda}\int_{-\infty}^{\tau} e^{\lambda\xi}\|g(\xi)\|^2 d\xi.$$

Therefore, there exists $T = T(\tau, D) > 2$ such that for all $t \geq T$ and $s \in [\tau-2, \tau]$,

$$e^{\lambda s}\|u(s, \tau-t, u_0(\tau-t))\|^2 \leq \frac{C}{\lambda}e^{\lambda\tau} + \frac{3}{\lambda}\int_{-\infty}^{\tau} e^{\lambda\xi}\|g(\xi)\|^2 d\xi. \tag{4.7}$$

Integrate the above with respect to s on $(\tau-2, \tau)$ to obtain that

$$\int_{\tau-2}^{\tau} e^{\lambda s}\|u(s, \tau-t, u_0(\tau-t))\|^2 ds \leq \frac{2C}{\lambda}e^{\lambda\tau} + \frac{6}{\lambda}\int_{-\infty}^{\tau} e^{\lambda\xi}\|g(\xi)\|^2 d\xi. \tag{4.8}$$

On the other hand, for $s = \tau - 2$, (4.7) implies that

$$e^{\lambda(\tau-2)}\|u(\tau-2, \tau-t, u_0(\tau-t))\|^2 \leq \frac{C}{\lambda}e^{\lambda\tau} + \frac{3}{\lambda}\int_{-\infty}^{\tau} e^{\lambda\xi}\|g(\xi)\|^2 d\xi. \tag{4.9}$$

Multiplying (4.4) by $e^{\lambda t}$ and then integrating over $(\tau-2, \tau)$, by (4.9) we get that, for all $t \geq T$,

$$\begin{aligned}
&e^{\lambda\tau}\|u(\tau, \tau-t, u_0(\tau-t))\|^2 + 2\int_{\tau-2}^{\tau} e^{\lambda\xi}\|\nabla u(\xi, \tau-t, u_0(\tau-t))\|^2 d\xi \\
&+2\alpha_1\int_{\tau-2}^{\tau} e^{\lambda\xi}\|u(\xi, \tau-t, u_0(\tau-t))\|_p^p d\xi \\
&\leq e^{\lambda(\tau-2)}\|u(\tau-2, \tau-t, u_0(\tau-t))\|^2 + \frac{2}{\lambda}\int_{\tau-2}^{\tau} e^{\lambda\xi}\|g(\xi)\|^2 d\xi + \frac{C}{\lambda}e^{\lambda\tau} \\
&\leq C\int_{-\infty}^{\tau} e^{\lambda\xi}\|g(\xi)\|^2 d\xi + Ce^{\lambda\tau},
\end{aligned}$$

which along with (4.8) completes the proof. □

Note that $e^{\lambda\xi} \geq e^{\lambda\tau - 2\lambda}$ for any $\xi \geq \tau - 2$. So as an immediate consequence of Lemma 4.2 we have the following estimates.

Corollary 4.3. *Suppose* (3.3) *and* (3.9) *hold. Then for every* $\tau \in \mathbb{R}$ *and* $D = \{D(t)\}_{t\in\mathbb{R}} \in \mathcal{D}_\lambda$, *there exists* $T = T(\tau, D) > 2$ *such that for all* $t \geq T$,

$$\int_{\tau-2}^{\tau} \|u(\xi, \tau-t, u_0(\tau-t))\|^2 d\xi \leq M + Me^{-\lambda\tau}\int_{-\infty}^{\tau} e^{\lambda\xi}\|g(\xi)\|^2 d\xi,$$

$$\int_{\tau-2}^{\tau} \|\nabla u(\xi, \tau-t, u_0(\tau-t))\|^2 d\xi \leq M + Me^{-\lambda\tau}\int_{-\infty}^{\tau} e^{\lambda\xi}\|g(\xi)\|^2 d\xi,$$

and

$$\int_{\tau-2}^{\tau} \|u(\xi, \tau-t, u_0(\tau-t))\|_p^p d\xi \leq M + Me^{-\lambda\tau}\int_{-\infty}^{\tau} e^{\lambda\xi}\|g(\xi)\|^2 d\xi,$$

where $u_0(\tau-t) \in D(\tau-t)$, *and* M *is a positive constant independent of* τ *and* D.

Next we derive uniform estimates of solutions in $H^1(\mathbb{R}^n)$.

Lemma 4.4. *Suppose* (3.3), (3.6) *and* (3.9) *hold. Then for every* $\tau \in \mathbb{R}$ *and* $D = \{D(t)\}_{t\in\mathbb{R}} \in \mathcal{D}_\lambda$, *there exists* $T = T(\tau, D) > 2$ *such that for all* $t \geq T$,

$$\|\nabla u(\tau, \tau - t, u_0(\tau - t))\|^2 \leq M + Me^{-\lambda\tau} \int_{-\infty}^{\tau} e^{\lambda\xi} \|g(\xi)\|^2 d\xi,$$

$$\|u(\tau, \tau - t, u_0(\tau - t))\|_p^p \leq M + Me^{-\lambda\tau} \int_{-\infty}^{\tau} e^{\lambda\xi} \|g(\xi)\|^2 d\xi,$$

and

$$\int_{\tau-1}^{\tau} \|u_\xi(\xi, \tau - t, u_0(\tau - t))\|^2 d\xi \leq M + Me^{-\lambda\tau} \int_{-\infty}^{\tau} e^{\lambda\xi} \|g(\xi)\|^2 d\xi,$$

where $u_0(\tau - t) \in D(\tau - t)$, *and* M *is a positive constant independent of* τ *and* D.

Proof. In the following, we write $u_0(\tau - t)$ as u_0 for convenience. Taking the inner product of (3.1) with u_t in $L^2(\mathbb{R}^n)$ and then replacing t by ξ, we obtain

$$\|u_\xi(\xi, \tau - t, u_0)\|^2 + \frac{d}{d\xi} \left(\frac{1}{2} \|\nabla u(\xi, \tau - t, u_0)\|^2 + \frac{1}{2} \lambda \|u(\xi, \tau - t, u_0)\|^2 - \int_{\mathbb{R}^n} F(x, u) dx \right)$$

$$= (g(\xi), u_\xi(\xi, \tau - t, u_0)).$$

Note that the right-hand side of the above is bounded by

$$|(g(\xi), u_\xi(\xi, \tau - t, u_0))| \leq \|g(\xi)\| \ \|u_\xi(\xi, \tau - t, u_0)\| \leq \frac{1}{2} \|u_\xi(\xi, \tau - t, u_0)\|^2 + \frac{1}{2} \|g(\xi)\|^2.$$

Then we have

$$\|u_\xi(\xi, \tau - t, u_0)\|^2 + \frac{d}{d\xi} \left(\|\nabla u(\xi, \tau - t, u_0)\|^2 + \lambda \|u(\xi, \tau - t, u_0)\|^2 - 2 \int_{\mathbb{R}^n} F(x, u) dx \right) \leq \|g(\xi)\|^2, \tag{4.10}$$

which implies that

$$\frac{d}{d\xi} \left(\|\nabla u(\xi, \tau - t, u_0)\|^2 + \lambda \|u(\xi, \tau - t, u_0)\|^2 - 2 \int_{\mathbb{R}^n} F(x, u) dx \right) \leq \|g(\xi)\|^2. \tag{4.11}$$

Let $s \leq \tau$ and $t \geq 2$. By integrating (4.11) over (s, τ) we get that

$$\|\nabla u(\tau, \tau - t, u_0)\|^2 + \lambda \|u(\tau, \tau - t, u_0)\|^2 - 2 \int_{\mathbb{R}^n} F(x, u(\tau, \tau - t, u_0)) dx$$

$$\leq \|\nabla u(s, \tau - t, u_0)\|^2 + \lambda \|u(s, \tau - t, u_0)\|^2 - 2 \int_{\mathbb{R}^n} F(x, u(s, \tau - t, u_0)) dx + \int_s^{\tau} \|g(\xi)\|^2 d\xi.$$

Now integrating the above with respect to s on $(\tau-1,\tau)$ we find that

$$\|\nabla u(\tau,\tau-t,u_0)\|^2+\lambda\|u(\tau,\tau-t,u_0)\|^2-2\int_{\mathbb{R}^n}F(x,u(\tau,\tau-t,u_0))dx$$
$$\le\int_{\tau-1}^{\tau}\|\nabla u(s,\tau-t,u_0)\|^2ds+\lambda\int_{\tau-1}^{\tau}\|u(s,\tau-t,u_0)\|^2ds$$
$$-2\int_{\tau-1}^{\tau}\int_{\mathbb{R}^n}F(x,u(s,\tau-t,u_0))dxds+\int_{\tau-1}^{\tau}\|g(\xi)\|^2d\xi. \tag{4.12}$$

By (3.6) we have

$$\alpha_5\|u(\tau,\tau-t,u_0(\tau-t))\|_p^p-\int_{\mathbb{R}^n}\phi_3(x)dx\le-\int_{\mathbb{R}^n}F(x,u(\tau,\tau-t,u_0))dx, \tag{4.13}$$

and

$$-\int_{\mathbb{R}^n}F(x,u(s,\tau-t,u_0))dx\le\alpha_4\|u(s,\tau-t,u_0(\tau-t))\|_p^p+\int_{\mathbb{R}^n}\phi_4(x)dx. \tag{4.14}$$

It follows from (4.12)-(4.14) that

$$\|\nabla u(\tau,\tau-t,u_0)\|^2+\lambda\|u(\tau,\tau-t,u_0)\|^2+2\alpha_5\|u(\tau,\tau-t,u_0)\|_p^p$$
$$\le\int_{\tau-1}^{\tau}\|\nabla u(s,\tau-t,u_0)\|^2ds+\lambda\int_{\tau-1}^{\tau}\|u(s,\tau-t,u_0)\|^2ds$$
$$+2\alpha_4\int_{\tau-1}^{\tau}\|u(s,\tau-t,u_0(\tau-t))\|_p^pds+\int_{\tau-1}^{\tau}\|g(\xi)\|^2d\xi+2\int_{\mathbb{R}^n}(\phi_3(x)+\phi_4(x))dx,$$

which along with Corollary 4.3 implies that there exists $T=T(\tau,D)>2$ such that for all $t\ge T$,

$$\|\nabla u(\tau,\tau-t,u_0)\|^2+\lambda\|u(\tau,\tau-t,u_0)\|^2+2\alpha_5\|u(\tau,\tau-t,u_0)\|_p^p$$
$$\le C+Ce^{-\lambda\tau}\int_{-\infty}^{\tau}e^{\lambda\xi}\|g(\xi)\|^2d\xi+\int_{\tau-1}^{\tau}\|g(\xi)\|^2d\xi$$
$$\le C+Ce^{-\lambda\tau}\int_{-\infty}^{\tau}e^{\lambda\xi}\|g(\xi)\|^2d\xi+e^{\lambda}e^{-\lambda\tau}\int_{-\infty}^{\tau}e^{\lambda\xi}\|g(\xi)\|^2d\xi. \tag{4.15}$$

Similarly, first integrating (4.11) with respect to ξ on $(s,\tau-1)$ and then integrating with respect to s on $(\tau-2,\tau-1)$, by using Corollary 4.3 we can get that for all $t\ge T$,

$$\|\nabla u(\tau-1,\tau-t,u_0)\|^2+\lambda\|u(\tau-1,\tau-t,u_0)\|^2+2\alpha_5\|u(\tau-1,\tau-t,u_0)\|_p^p$$
$$\le C+Ce^{-\lambda\tau}\int_{-\infty}^{\tau}e^{\lambda\xi}\|g(\xi)\|^2d\xi+\int_{\tau-2}^{\tau-1}\|g(\xi)\|^2d\xi. \tag{4.16}$$

Now integrating (4.10) over $(\tau-1,\tau)$ we obtain that

$$\int_{\tau-1}^{\tau}\|u_\xi(\xi,\tau-t,u_0)\|^2d\xi+\|\nabla u(\tau,\tau-t,u_0)\|^2+\lambda\|u(\tau,\tau-t,u_0)\|^2-2\int_{\mathbb{R}^n}F(x,u(\tau))dx$$

$$\leq \int_{\tau-1}^{\tau} \|g(\xi)\|^2 d\xi + \|\nabla u(\tau-1, \tau-t, u_0)\|^2 + \lambda \|u(\tau-1, \tau-t, u_0)\|^2 - 2\int_{\mathbb{R}^n} F(x, u(\tau-1))dx,$$

which along with (4.13), (4.14) and (4.16) shows that for all $t \geq T$,

$$\int_{\tau-1}^{\tau} \|u_\xi(\xi, \tau-t, u_0)\|^2 d\xi \leq \int_{\tau-1}^{\tau} \|g(\xi)\|^2 d\xi + 2\int_{\mathbb{R}^n} (\phi_3(x) + \phi_4(x))dx$$

$$+\|\nabla u(\tau-1, \tau-t, u_0)\|^2 + \lambda \|u(\tau-1, \tau-t, u_0)\|^2 + 2\alpha_4 \|u(\tau-1, \tau-t, u_0)\|_p^p$$

$$\leq C + Ce^{-\lambda\tau} \int_{-\infty}^{\tau} e^{\lambda\xi} \|g(\xi)\|^2 d\xi + \int_{\tau-2}^{\tau} \|g(\xi)\|^2 d\xi$$

$$\leq C + Ce^{-\lambda\tau} \int_{-\infty}^{\tau} e^{\lambda\xi} \|g(\xi)\|^2 d\xi + e^{2\lambda} e^{-\lambda\tau} \int_{-\infty}^{\tau} e^{\lambda\xi} \|g(\xi)\|^2 d\xi. \tag{4.17}$$

Then Lemma 4.4 follows from (4.15) and (4.17) immediately. □

We now derive uniform estimates of the derivatives of solutions in time. To this end, we also assume $\frac{dg}{dt} \in L^2_{loc}(\mathbb{R}, L^2(\mathbb{R}^n))$.

Lemma 4.5. *Suppose* (3.3)-(3.6) *and* (3.9) *hold. Let* $\frac{dg}{dt} \in L^2_{loc}(\mathbb{R}, L^2(\mathbb{R}^n))$. *Then for every* $\tau \in \mathbb{R}$ *and* $D = \{D(t)\}_{t\in\mathbb{R}} \in \mathcal{D}_\lambda$, *there exists* $T = T(\tau, D) > 2$ *such that for all* $t \geq T$,

$$\|u_\tau(\tau, \tau-t, u_0(\tau-t))\|^2 \leq M + Me^{-\lambda\tau} \int_{-\infty}^{\tau} e^{\lambda\xi} \|g(\xi)\|^2 d\xi + M \int_{\tau-1}^{\tau} \|g_\xi(\xi)\|^2 d\xi,$$

where $u_0(\tau-t) \in D(\tau-t)$, *and* M *is a positive constant independent of* τ *and* D.

Proof. Let $u_t = v$ and differentiate (3.1) with respect to t to get that

$$\frac{\partial v}{\partial t} - \Delta v + \lambda v = \frac{\partial f}{\partial u}(x, u)v + g_t(x, t).$$

Taking the inner product of the above with v in $L^2(\mathbb{R}^n)$, we obtain

$$\frac{1}{2}\frac{d}{dt}\|v\|^2 + \|\nabla v\|^2 + \lambda\|v\|^2 = \int_{\mathbb{R}^n} \frac{\partial f}{\partial u}(x, u)|v(x, t)|^2 dx + \int_{\mathbb{R}^n} g_t(x, t)v(x, t)dx. \tag{4.18}$$

By (3.5) and the Young inequality, it follows from (4.18) that

$$\frac{d}{dt}\|v\|^2 \leq 2\alpha_3\|v\|^2 + \frac{1}{\lambda}\|g_t(t)\|^2. \tag{4.19}$$

Let $s \in [\tau-1, \tau]$ and $t \geq 1$. Integrating (4.19) on (s, τ), by $v = u_t$ we get that

$$\|u_\tau(\tau, \tau-t, u_0(\tau-t))\|^2 \leq \|u_s(s, \tau-t, u_0(\tau-t))\|^2$$

$$+2\alpha_3 \int_s^\tau \|u_\xi(\xi, \tau - t, u_0(\tau - t))\|^2 d\xi + \frac{1}{\lambda} \int_s^\tau \|g_\xi(\xi)\|^2 d\xi.$$

$$\leq \|u_s(s, \tau - t, u_0(\tau - t))\|^2 + 2\alpha_3 \int_{\tau-1}^\tau \|u_\xi(\xi, \tau - t, u_0(\tau - t))\|^2 d\xi + \frac{1}{\lambda} \int_{\tau-1}^\tau \|g_\xi(\xi)\|^2 d\xi.$$

Now integrating the above with respect to s on $(\tau - 1, \tau)$ we find that

$$\|u_\tau(\tau, \tau - t, u_0(\tau - t))\|^2 \leq \int_{\tau-1}^\tau \|u_s(s, \tau - t, u_0(\tau - t))\|^2 ds$$

$$+2\alpha_3 \int_{\tau-1}^\tau \|u_\xi(\xi, \tau - t, u_0(\tau - t))\|^2 d\xi + \frac{1}{\lambda} \int_{\tau-1}^\tau \|g_\xi(\xi)\|^2 d\xi,$$

which along with Lemma 4.4 shows that there exists $T = (\tau, D) > 2$ such that for all $t \geq T$,

$$\|u_\tau(\tau, \tau - t, u_0(\tau - t))\|^2$$

$$\leq C + Ce^{-\lambda\tau} \int_{-\infty}^\tau e^{\lambda\xi} \|g(\xi)\|^2 d\xi + \frac{1}{\lambda} \int_{\tau-1}^\tau \|g_\xi(\xi)\|^2 d\xi.$$

The proof is completed. $\square$

We now establish uniform estimates on the tails of solutions when $t \to \infty$. We show that the tails of solutions are uniformly small for large space and time variables. These uniform estimates are crucial for proving the pullback asymptotic compactness of the cocycle ϕ.

Lemma 4.6. *Suppose* (3.3), (3.6) *and* (3.9)-(3.10) *hold. Then for every $\eta > 0$, $\tau \in \mathbb{R}$ and $D = \{D(t)\}_{t\in\mathbb{R}} \in \mathcal{D}_\lambda$, there exists $T = T(\tau, D, \eta) > 2$ and $K = K(\tau, \eta) > 0$ such that for all $t \geq T$ and $k \geq K$,*

$$\int_{|x|\geq k} |u(x, \tau, \tau - t, u_0(\tau - t))|^2 dx \leq \eta,$$

where $u_0(\tau - t) \in D(\tau - t)$, $K(\tau, \eta)$ depends on τ and η, and $T(\tau, D, \eta)$ depends on τ, D and η.

Proof. We use a cut-off technique to establish the estimates on the tails of solutions. Let θ be a smooth function satisfying $0 \leq \theta(s) \leq 1$ for $s \in \mathbb{R}^+$, and

$$\theta(s) = 0 \text{ for } 0 \leq s \leq 1; \quad \theta(s) = 1 \text{ for } s \geq 2.$$

Then there exists a constant C such that $|\theta'(s)| \leq C$ for $s \in \mathbb{R}^+$. Taking the inner product of (3.1) with $\theta(\frac{|x|^2}{k^2})u$ in $L^2(\mathbb{R}^n)$, we get

$$\frac{1}{2}\frac{d}{dt} \int_{\mathbb{R}^n} \theta(\frac{|x|^2}{k^2})|u|^2 - \int_{\mathbb{R}^n} \theta(\frac{|x|^2}{k^2}) u\Delta u + \lambda \int_{\mathbb{R}^n} \theta(\frac{|x|^2}{k^2})|u|^2$$

$$= \int_{\mathbb{R}^n} \theta(\frac{|x|^2}{k^2}) f(x,u) u dx + \int_{\mathbb{R}^n} \theta(\frac{|x|^2}{k^2}) g(x,t) u(x,t) dx. \tag{4.20}$$

We now estimate the right-hand side of (4.20). For the nonlinear term, by (3.3) we have

$$\int_{\mathbb{R}^n} \theta(\frac{|x|^2}{k^2}) f(x,u) u dx \le -\alpha_1 \int_{\mathbb{R}^n} \theta(\frac{|x|^2}{k^2}) |u|^p dx + \int_{\mathbb{R}^n} \theta(\frac{|x|^2}{k^2}) \phi_1(x) dx \le \int_{|x|\ge k} \phi_1(x) dx. \tag{4.21}$$

For the last term on the right-hand side of (4.20) we find that

$$\int_{\mathbb{R}^n} \theta(\frac{|x|^2}{k^2}) g(x,t) u(x,t) dx = \int_{|x|\ge k} \theta(\frac{|x|^2}{k^2}) g(x,t) u(x,t) dx$$

$$\le \frac{1}{2}\lambda \int_{|x|\ge k} \theta^2(\frac{|x|^2}{k^2}) |u|^2 dx + \frac{1}{2\lambda} \int_{|x|\ge k} |g(x,t)|^2 dx$$

$$\le \frac{1}{2}\lambda \int_{\mathbb{R}^n} \theta(\frac{|x|^2}{k^2}) |u|^2 dx + \frac{1}{2\lambda} \int_{|x|\ge k} |g(x,t)|^2 dx. \tag{4.22}$$

On the other hand, for the second term on the left-hand side of (4.20), by integration by parts, we have

$$\int_{\mathbb{R}^n} \theta(\frac{|x|^2}{k^2}) u \Delta u = -\int_{\mathbb{R}^n} \theta(\frac{|x|^2}{k^2}) |\nabla u|^2 - \int_{\mathbb{R}^n} \theta'(\frac{|x|^2}{k^2})(\frac{2x}{k^2} \cdot \nabla u) u.$$

$$\le -\int_{k\le|x|\le\sqrt{2}k} \theta'(\frac{|x|^2}{k^2})(\frac{2x}{k^2} \cdot \nabla u) u \le \frac{C}{k} \int_{k\le|x|\le\sqrt{2}k} |u||\nabla u| \le \frac{C}{k}(\|u\|^2 + \|\nabla u\|^2), \tag{4.23}$$

where C is independent of k. It follows from (4.20)-(4.23) that

$$\frac{d}{dt} \int_{\mathbb{R}^n} \theta(\frac{|x|^2}{k^2}) |u|^2 dx + \lambda \int_{\mathbb{R}^n} \theta(\frac{|x|^2}{k^2}) |u|^2 dx$$

$$\le 2\int_{|x|\ge k} |\phi_1(x)| dx + \frac{1}{\lambda} \int_{|x|\ge k} |g(x,t)|^2 dx + \frac{C}{k}(\|u\|^2 + \|\nabla u\|^2). \tag{4.24}$$

Multiplying (4.24) by $e^{\lambda t}$ and then integrating over $(\tau - t, \tau)$ with $t \ge 0$, we get that

$$\int_{\mathbb{R}^n} \theta(\frac{|x|^2}{k^2}) |u(x,\tau,\tau-t,u_0(\tau-t))|^2 dx$$

$$\le e^{-\lambda\tau} e^{\lambda(\tau-t)} \int_{\mathbb{R}^n} \theta(\frac{|x|^2}{k^2}) |u_0(x,\tau-t)|^2 dx$$

$$+2e^{-\lambda\tau} \int_{\tau-t}^{\tau} \int_{|x|\ge k} e^{\lambda\xi} |\phi_1(x)| dx d\xi + \frac{1}{\lambda} e^{-\lambda\tau} \int_{\tau-t}^{\tau} \int_{|x|\ge k} e^{\lambda\xi} |g(x,\xi)|^2 dx d\xi$$

$$+\frac{C}{k} e^{-\lambda\tau} \int_{\tau-t}^{\tau} e^{\lambda\xi} \left(\|u(\xi,\tau-t,u_0(\tau-t))\|^2 + \|\nabla u(\xi,\tau-t,u_0(\tau-t))\|^2 \right) d\xi$$

$$\le e^{-\lambda\tau} e^{\lambda(\tau-t)} \int_{\mathbb{R}^n} |u_0(x,\tau-t)|^2 dx$$

$$+2e^{-\lambda\tau}\int_{-\infty}^{\tau}\int_{|x|\geq k} e^{\lambda\xi}|\phi_1(x)|dxd\xi + \frac{1}{\lambda}e^{-\lambda\tau}\int_{-\infty}^{\tau}\int_{|x|\geq k} e^{\lambda\xi}|g(x,\xi)|^2 dxd\xi$$

$$+\frac{C}{k}e^{-\lambda\tau}\int_{\tau-t}^{\tau} e^{\lambda\xi}\left(\|u(\xi,\tau-t,u_0(\tau-t))\|^2+\|\nabla u(\xi,\tau-t,u_0(\tau-t))\|^2\right)d\xi. \tag{4.25}$$

Note that given $\eta>0$, there is $T_1=T_1(\tau,D,\eta)>0$ such that for all $t\geq T_1$,

$$e^{-\lambda\tau}e^{\lambda(\tau-t)}\int_{|R^n} |u_0(\tau-t)|^2 dx \leq \eta. \tag{4.26}$$

Since $\phi_1\in L^1(\mathbb{R}^n)$, there exists $K_1=K_1(\eta)>0$ such that for all $k\geq K_1$,

$$2e^{-\lambda\tau}\int_{-\infty}^{\tau}\int_{|x|\geq k} e^{\lambda\xi}|\phi_1(x)|dxd\xi \leq \eta. \tag{4.27}$$

On the other hand, by (3.11) there is $K_2=K_2(\tau,\eta)>K_1$ such that for all $k\geq K_2$,

$$\frac{1}{\lambda}e^{-\lambda\tau}\int_{-\infty}^{\tau}\int_{|x|\geq k} e^{\lambda\xi}|g(x,\xi)|^2 dxd\xi \leq \frac{\eta}{\lambda}. \tag{4.28}$$

For the last term on the right-hand side of (4.25), it follows from Lemma 4.1 that there is $T_2=T_2(\tau,D)>0$ such that for all $t\geq T_2$,

$$\frac{C}{k}e^{-\lambda\tau}\int_{\tau-t}^{\tau} e^{\lambda\xi}\left(\|u(\xi,\tau-t,u_0(\tau-t))\|^2+\|\nabla u(\xi,\tau-t,u_0(\tau-t))\|^2\right)d\xi$$

$$\leq \frac{C}{k}\left(1+e^{-\lambda\tau}\right)\int_{-\infty}^{\tau} e^{\lambda\xi}\|g(\xi)\|^2 d\xi.$$

Therefore, there is $K_3=K_3(\tau,\eta)>K_2$ such that for all $k\geq K_3$ and $t\geq T_2$,

$$\frac{C}{k}e^{-\lambda\tau}\int_{\tau-t}^{\tau} e^{\lambda\xi}\left(\|u(\xi,\tau-t,u_0(\tau-t))\|^2+\|\nabla u(\xi,\tau-t,u_0(\tau-t))\|^2\right)d\xi \leq \eta. \tag{4.29}$$

Let $T=\max\{T_1,T_2\}$. Then by (4.25)-(4.29) we find that for all $k\geq K_3$ and $t\geq T$,

$$\int_{\mathbb{R}^n}\theta(\frac{|x|^2}{k^2})|u(x,\tau,\tau-t,u_0(\tau-t))|^2 dx \leq 3\eta+\frac{\eta}{\lambda},$$

and hence for all $k\geq K_3$ and $t\geq T$,

$$\int_{|x|\geq\sqrt{2}k}|u(x,\tau,\tau-t,u_0(\tau-t))|^2 dx$$

$$\leq \int_{\mathbb{R}^n}\theta(\frac{|x|^2}{k^2})|u(x,\tau,\tau-t,u_0(\tau-t))|^2 dx \leq 3\eta+\frac{\eta}{\lambda},$$

which completes the proof. □

5 Existence of pullback attractors

In this section, we prove the existence of a $\mathcal{D}_\lambda$-pullback global attractor for the non-autonomous Reaction-Diffusion equation on $\mathbb{R}^n$. We first establish the $\mathcal{D}_\lambda$-pullback asymptotic compactness of solutions and prove the existence of a pullback attractor in $L^2(\mathbb{R}^n)$. Then we show that this attractor is actually a $\mathcal{D}_\lambda$-pullback attractor in $H^1(\mathbb{R}^n)$.

Lemma 5.1. *Suppose* (3.3)-(3.6) *and* (3.9)-(3.10) *hold. Then ϕ is $\mathcal{D}_\lambda$-pullback asymptotically compact in $L^2(\mathbb{R}^n)$, that is, for every $\tau \in \mathbb{R}$, $D = \{D(t)\}_{t\in\mathbb{R}} \in \mathcal{D}_\lambda$, and $t_n \to \infty$, $u_{0,n} \in D(\tau - t_n)$, the sequence $\phi(t_n, \tau - t_n, u_{0,n})$ has a convergent subsequence in $L^2(\mathbb{R}^n)$.*

Proof. We use the uniform estimates on the tails of solutions to establish the precompactness of $\phi(t_n, \tau - t_n, u_{0,n})$ in $L^2(\mathbb{R}^n)$, that is, we prove that for every $\eta > 0$, the sequence $\phi(t_n, \tau - t_n, u_{0,n})$ has a finite covering of balls of radii less than η. Given $K > 0$, denote by

$$\Omega_K = \{x : |x| \le K\} \quad \text{and} \quad \Omega_K^c = \{x : |x| > K\}.$$

Then by Lemma 4.6, for the given $\eta > 0$, there exist $K = K(\tau, \eta) > 0$ and $T = T(\tau, D, \eta) > 2$ such that for $t \ge T$,

$$\|\phi(t, \tau - t, u_0(\tau - t))\|_{L^2(\Omega_K^c)} \le \frac{\eta}{4}.$$

Since $t_n \to \infty$, there is $N_1 = N_1(\tau, D, \eta) > 0$ such that $t_n \ge T$ for all $n \ge N_1$, and hence we obtain that, for all $n \ge N_1$,

$$\|\phi(t_n, \tau - t_n, u_{0,n})\|_{L^2(\Omega_K^c)} \le \frac{\eta}{4}. \tag{5.1}$$

On the other hand, by Lemmas 4.1 and 4.4, there exist $C = C(\tau) > 0$ and $N_2(\tau, D) > 0$ such that for all $n \ge N_2$,

$$\|\phi(t_n, \tau - t_n, u_{0,n})\|_{H^1(\Omega_K)} \le C. \tag{5.2}$$

By the compactness of embedding $H^1(\Omega_K) \hookrightarrow L^2(\Omega_K)$, the sequence $\phi(t_n, \tau - t_n, u_{0,n})$ is precompact in $L^2(\Omega_K)$. Therefore, for the given $\eta > 0$, $\phi(t_n, \tau - t_n, u_{0,n})$ has a finite covering in $L^2(\Omega_K)$ of balls of radii less than $\eta/4$, which along with (5.1) shows that $\phi(t_n, \tau - t_n, u_{0,n})$ has a finite covering in $L^2(\mathbb{R}^n)$ of balls of radii less than η, and thus $\phi(t_n, \tau - t_n, u_{0,n})$ is precompact in $L^2(\mathbb{R}^n)$. □

We now present the existence of a pullback global attractor for ϕ in $L^2(\mathbb{R}^n)$.

Theorem 5.2. *Suppose* (3.3)-(3.6) *and* (3.9)-(3.10) *hold. Then problem* (3.1)-(3.2) *has a unique* $\mathcal{D}_\lambda$*-pullback global attractor* $\{\mathcal{A}(\tau)\}_{\tau\in\mathbb{R}} \in \mathcal{D}_\lambda$ *in* $L^2(\mathbb{R}^n)$*, that is, for every* $\tau \in \mathbb{R}$,

(i) $\mathcal{A}(\tau)$ *is compact in* $L^2(\mathbb{R}^n)$*;*

(ii) $\{\mathcal{A}(\tau)\}_{\tau\in\mathbb{R}}$ *is invariant, that is,*

$$\phi(t, \tau, \mathcal{A}(\tau)) = \mathcal{A}(t + \tau), \quad \forall\ t \geq 0;$$

(iii) $\{\mathcal{A}(\tau)\}_{\tau\in\mathbb{R}}$ *attracts every set in* $\mathcal{D}_\lambda$ *with respect to the norm of* $L^2(\mathbb{R}^n)$*, that is, for every* $B = \{B(\tau)\}_{\tau\in\mathbb{R}} \in \mathcal{D}_\lambda$,

$$\lim_{t\to\infty} d_{L^2(\mathbb{R}^n)}(\phi(t, \tau - t, B(\tau - t)), \mathcal{A}(\tau)) = 0,$$

where for any $Y,\ Z \subseteq L^2(\mathbb{R}^n)$,

$$d_{L^2(\mathbb{R}^n)}(Y, Z) = \sup_{y\in Y} \inf_{z\in Z} \|y - z\|_{L^2(\mathbb{R}^n)}.$$

Proof. For $\tau \in \mathbb{R}$, denote by

$$B(\tau) = \{u :\ \|u\|^2 \leq M + Me^{-\lambda\tau} \int_{-\infty}^{\tau} e^{\lambda\xi} \|g(\xi)\|^2 d\xi\},$$

where M is the positive constant in Lemma 4.1. Note that $B = \{B(\tau)\}_{\tau\in\mathbb{R}} \in \mathcal{D}_\lambda$ is a $\mathcal{D}_\lambda$-pullback absorbing for ϕ in $L^2(\mathbb{R}^n)$ by Lemma 4.1. In addition, ϕ is $\mathcal{D}_\lambda$-pullback asymptotically compact by Lemma 5.1. Thus the existence of a $\mathcal{D}_\lambda$-pullback global attractor for ϕ in $L^2(\mathbb{R}^n)$ follows from Proposition 2.7. □

In what follows, we strengthen Theorem 5.2 and show that the global attractor $\{\mathcal{A}(\tau)\}_{\tau\in\mathbb{R}}$ is actually a $\mathcal{D}_\lambda$-pullback global attractor in $H^1(\mathbb{R}^n)$. As a necessary step towards this goal, we first prove the asymptotic compactness of solutions in $H^1(\mathbb{R}^n)$.

Lemma 5.3. *Suppose* (3.3)-(3.6) *and* (3.9)-(3.10) *hold. Let* $\frac{dg}{dt} \in L^2_{loc}(\mathbb{R}, L^2(\mathbb{R}^n))$*. Then* ϕ *is* $\mathcal{D}_\lambda$*-pullback asymptotically compact in* $H^1(\mathbb{R}^n)$*, that is, for every* $\tau \in \mathbb{R}$, $D = \{D(t)\}_{t\in\mathbb{R}} \in \mathcal{D}_\lambda$*, and* $t_n \to \infty$, $u_{0,n} \in D(\tau - t_n)$*, the sequence* $\phi(t_n, \tau - t_n, u_{0,n})$ *has a convergent subsequence in* $H^1(\mathbb{R}^n)$.

Proof. By Lemma 5.1, the sequence $\phi(t_n, \tau - t_n, u_{0,n}) = u(\tau, \tau - t_n, u_{0,n})$ has a convergent subsequence in $L^2(\mathbb{R}^n)$, and hence there exists $v \in L^2(\mathbb{R}^n)$ such that, up to a subsequence,

$$u(\tau, \tau - t_n, u_{0,n}) \to v \quad \text{in} \quad L^2(\mathbb{R}^n).$$

This shows that

$$u(\tau, \tau - t_n, u_{0,n}) \quad \text{is a Cauchy sequence in} \quad L^2(\mathbb{R}^n). \tag{5.3}$$

Next we prove that $u(\tau, \tau - t_n, u_{0,n})$ is actually a Cauchy sequence in $H^1(\mathbb{R}^n)$. For any $n, m \geq 1$, it follows from (3.1) that

$$-\Delta \left(u(\tau, \tau - t_n, u_{0,n}) - u(\tau, \tau - t_m, u_{0,m}) \right) + \lambda \left(u(\tau, \tau - t_n, u_{0,n}) - u(\tau, \tau - t_m, u_{0,m}) \right)$$

$$= f(x, u(\tau, \tau - t_n, u_{0,n})) - f(x, u(\tau, \tau - t_m, u_{0,m})) - u_\tau(\tau, \tau - t_n, u_{0,n}) + u_\tau(\tau, \tau - t_m, u_{0,m}). \tag{5.4}$$

Multiplying (5.4) by $u(\tau, \tau - t_n, u_{0,n}) - u(\tau, \tau - t_m, u_{0,m})$, by (3.5) we get that

$$\|\nabla \left(u(\tau, \tau - t_n, u_{0,n}) - u(\tau, \tau - t_m, u_{0,m}) \right) \|^2 + \lambda \| u(\tau, \tau - t_n, u_{0,n}) - u(\tau, \tau - t_m, u_{0,m}) \|^2$$

$$\leq \| u_\tau(\tau, \tau - t_n, u_{0,n}) - u_\tau(\tau, \tau - t_m, u_{0,m}) \| \| u(\tau, \tau - t_n, u_{0,n}) - u(\tau, \tau - t_m, u_{0,m}) \|.$$

$$+ \alpha_3 \| u(\tau, \tau - t_n, u_{0,n}) - u(\tau, \tau - t_m, u_{0,m}) \|^2. \tag{5.5}$$

By Lemma 4.5 we find that for every $\tau \in \mathbb{R}$, there exists $T = T(\tau, D)$ such that for all $t \geq T$,

$$\| u_\tau(\tau, \tau - t, u_0(\tau - t)) \| \leq C.$$

Since $t_n \to \infty$, there exists $N = N(\tau, D)$ such that $t_n \geq T$ for all $n \geq T$. Thus we obtain that, for all $n \geq N$,

$$\| u_\tau(\tau, \tau - t_n, u_{0,n}) \| \leq C,$$

which along with (5.5) shows that, for all $n, m \geq N$,

$$\|\nabla \left(u(\tau, \tau - t_n, u_{0,n}) - u(\tau, \tau - t_m, u_{0,m}) \right) \|^2 + \lambda \| u(\tau, \tau - t_n, u_{0,n}) - u(\tau, \tau - t_m, u_{0,m}) \|^2$$

$$\leq 2C \| u(\tau, \tau - t_n, u_{0,n}) - u(\tau, \tau - t_m, u_{0,m}) \|.$$

$$+ \alpha_3 \| u(\tau, \tau - t_n, u_{0,n}) - u(\tau, \tau - t_m, u_{0,m}) \|^2. \tag{5.6}$$

It follows from (5.3) and (5.6) that $u(\tau, \tau - t_n, u_{0,n})$ is a Cauchy sequence in $H^1(\mathbb{R}^n)$. The proof is completed. $\square$

We are now ready to prove the existence of a global attractor for problem (3.1)-(3.2) in $H^1(\mathbb{R}^n)$.

Theorem 5.4. *Suppose* (3.3)-(3.6) *and* (3.9)-(3.10) *hold. Let* $\frac{dg}{dt} \in L^2_{loc}(\mathbb{R}, L^2(\mathbb{R}^n))$. *Then problem* (3.1)-(3.2) *has a unique* $\mathcal{D}_\lambda$*-pullback global attractor* $\{\mathcal{A}(\tau)\}_{\tau\in\mathbb{R}} \in \mathcal{D}_\lambda$ *in* $H^1(\mathbb{R}^n)$*, that is, for every* $\tau \in \mathbb{R}$*,*

(i) $\mathcal{A}(\tau)$ *is compact in* $H^1(\mathbb{R}^n)$*;*

(ii) $\{\mathcal{A}(\tau)\}_{\tau\in\mathbb{R}}$ *is invariant, that is,*

$$\phi(t, \tau, \mathcal{A}(\tau)) = \mathcal{A}(t + \tau), \quad \forall\ t \geq 0;$$

(iii) $\{\mathcal{A}(\tau)\}_{\tau\in\mathbb{R}}$ *attracts every set in* $\mathcal{D}_\lambda$ *with respect to the norm of* $H^1(\mathbb{R}^n)$*, that is, for every* $B = \{B(\tau)\}_{\tau\in\mathbb{R}} \in \mathcal{D}_\lambda$*,*

$$\lim_{t\to\infty} d_{H^1(\mathbb{R}^n)}(\phi(t, \tau - t, B(\tau - t)), \mathcal{A}(\tau)) = 0,$$

where for any $Y,\ Z \subseteq H^1(\mathbb{R}^n)$*,*

$$d_{H^1(\mathbb{R}^n)}(Y, Z) = \sup_{y\in Y} \inf_{z\in Z} \|y - z\|_{H^1(\mathbb{R}^n)}.$$

Proof. The invariance of $\{\mathcal{A}(\tau)\}_{\tau\in\mathbb{R}}$ is already given in Theorem 5.2. So we only need to prove (i) and (iii).

Proof of (i). Let $\{v_n\}_{n=1}^\infty \subseteq \mathcal{A}(\tau)$. We want to show that there exists $v \in \mathcal{A}(\tau)$ such that, up to a subsequence, $v_n \to v$ in $H^1(\mathbb{R}^n)$. Since $\mathcal{A}(\tau)$ is compact in $L^2(\mathbb{R}^n)$ by Theorem 5.2, there exists $v \in \mathcal{A}(\tau)$ such that, up to a subsequence,

$$v_n \to v \quad \text{in} \quad L^2(\mathbb{R}^n). \tag{5.7}$$

We now prove the convergence in (5.7) actually holds in $H^1(\mathbb{R}^n)$. Let $\{t_n\}_{n=1}^\infty$ be a sequence with $t_n \to \infty$. By the invariance of $\{\mathcal{A}(\tau)\}_{\tau\in\mathbb{R}}$, for every $n \geq 1$, there exists $w_n \in \mathcal{A}(\tau - t_n)$ such that

$$v_n = \phi(t_n, \tau - t_n, w_n). \tag{5.8}$$

By Lemma 5.3, it follows from (5.8) that, there exist $\tilde{v} \in H^1(\mathbb{R}^n)$ such that, up to a subsequence,

$$v_n = \phi(t_n, \tau - t_n, w_n) \to \tilde{v} \quad \text{in} \quad H^1(\mathbb{R}^n). \tag{5.9}$$

Notice that (5.7) and (5.9) imply $\tilde{v} = v \in \mathcal{A}(\tau)$, and thus (i) follows.

Proof of (iii). Suppose (iii) is not true. Then there are $\tau \in \mathbb{R}$, $B = \{B(\tau)\}_{\tau\in\mathbb{R}} \in \mathcal{D}_\lambda$, $\epsilon_0 > 0$ and $t_n \to \infty$ such that

$$d_{H^1(\mathbb{R}^n)}(\phi(t_n, \tau - t_n, B(\tau - t_n)), \mathcal{A}(\tau)) \geq 2\epsilon_0,$$

which implies that for every $n \geq 1$, there exists $v_n \in B(\tau - t_n)$ such that

$$d_{H^1(\mathbb{R}^n)}(\phi(t_n, \tau - t_n, v_n), \mathcal{A}(\tau)) \geq \epsilon_0. \tag{5.10}$$

On the other hand, By Lemma 5.3, there is $v \in H^1(\mathbb{R}^n)$ such that, up to a subsequence,

$$\phi(t_n, \tau - t_n, v_n) \to v \quad \text{in} \quad H^1(\mathbb{R}^n). \tag{5.11}$$

Since $\{\mathcal{A}(\tau)\}_{\tau \in \mathbb{R}}$ attracts $B = \{B(\tau)\}_{\tau \in \mathbb{R}}$ in $L^2(\mathbb{R}^n)$ by Theorem 5.2, we have

$$\lim_{n \to \infty} d_{L^2(\mathbb{R}^n)}(\phi(t_n, \tau - t_n, v_n), \mathcal{A}(\tau)) = 0. \tag{5.12}$$

By (5.11)-(5.12) and the compactness of $\mathcal{A}(\tau)$, we find that $v \in \mathcal{A}(\tau)$ and

$$\lim_{n \to \infty} d_{H^1(\mathbb{R}^n)}(\phi(t_n, \tau - t_n, v_n), \mathcal{A}(\tau)) \leq \lim_{n \to \infty} d_{H^1(\mathbb{R}^n)}(\phi(t_n, \tau - t_n, v_n), v) = 0, \tag{5.13}$$

a contradiction with (5.10). The proof is completed. $\square$

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